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Note

Shock Wave Propagation in an Inhomogeneous Medium Using Finite Differences

INTRODUCTION

Finite-difference fluid equations have been known [1–4] to provide valid solutions to problems containing shocks when the physically correct conservation variables in conservation form are used. In this note, we examine the problem of shock propagation in an inhomogeneous medium with exponentially varying density. The problem and its analytical solution for both the increasing and decreasing density cases are well described by Zeldovich and Raizer [5]. This one-dimensional self-similar analytic solution will be compared to various numerical solutions using different algorithms. This problem constitutes a more severe numerical test for shock propagation than the shock wave in an homogeneous medium and therefore serves as a better test of numerical algorithms.

In particular we find that care must be taken in the use of forms of the hydrodynamic equations which do not express physical conservation. For nonconservation formulations of the energy equation an artificial viscosity must be introduced [6], not only to provide the necessary stability, but also to provide shock heating. The magnitude of this artificial viscosity to obtain best shock results depends on the grid size and the problem type. There is no simple way to obtain this optimal viscosity for problems where the solution is not known in advance.

In addition, we find that flux-corrected transport [7] (FCT) has several properties which make it more flexible and effective for shock calculations. While the comparison between different energy formulations has been made easier through the use of the FCT scheme, the results hold for any finite-difference algorithm and in particular they will be shown to hold using the Lax–Wendroff scheme as well.

Three different variables have been used for the energy equation, while the continuity and momentum equations have been treated in their usual conservative form. The numerical algorithms used treat all conservative terms in conservative finite-difference form. Use has been made of a Lax–Wendroff scheme (LW) and a flux-corrected transport (FCT) scheme. The Lax–Wendroff scheme uses the two-step Richtmeyer form coupled with a Von Neumann viscosity to provide the additional stability and viscous heating needed near the shocks. The FCT scheme makes use of artificial viscosity (Von Neumann type) when the energy

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equation is cast in terms of the pressure or temperature variable, to provide the viscous heating otherwise lacking in the shocks. Artificial viscosity is not needed in the strict conservative formulation.

The total energy, pressure and temperature equations are respectively:

$$(\partial E/\partial t) + \nabla \cdot \mathbf{v}E = -\nabla \cdot (p + q)\mathbf{v} \tag{1}$$

$$(\partial p/\partial t) + \nabla \cdot \mathbf{v}p = -(\gamma - 1)(p + q)\nabla \cdot \mathbf{v} \tag{2}$$

$$(\partial T/\partial t) + \nabla \cdot \mathbf{v}T = -[(\gamma - 2)T + (\gamma - 1)(q/\rho)]\nabla \cdot \mathbf{v} \tag{3}$$

$$q = -\rho b(\delta x^2) \mid \partial u/\partial x \mid \partial u/\partial x \tag{4}$$

where $E = \rho(\epsilon + \frac{1}{2}v^2)$ and $\epsilon = p/(\gamma - 1)\rho$ for a perfect gas.

The failure to compute accurately the dissipation mechanism which converts kinetic energy to thermal energy in a shock leads to a failure to conserve energy in the temperature and pressure formulations and hence gives incorrect results for the shock dynamics. Since the total energy equation is in divergence form whether the viscosity terms are included or not, conservation of energy is automatically guaranteed when a conservative difference scheme is applied. In the remainder of this note we will show the results of several test calculations demonstrating this. Section 2 shows the results for a shock propagating into an exponentially increasing density medium. The results of the different formulations are compared for several values of the artificial viscosity parameters and grid sizes. In Section 3 the results for the decreasing density case are shown and in Section 4 the conclusions that can be drawn from this study are made.

Results for Increasing Density Case

For a $\gamma = 2$ gas the self-similar solution is completely analytic [5]. From the Rankine–Hugoniot relations, we expect the density jump across the shock to be equal to 3. Figure 1 shows density profiles for the energy and temperature equations respectively for the LW and FCT schemes after a time $t = 350\ \delta t$. At that time, the shock has moved over a distance equal to 1.3Δ where Δ is the characteristic length, or scale height of the medium. Only one value of the viscosity coefficient is shown for illustrative purposes. The effect of nonconservation is shown clearly. In the LW total energy formulation, the value of the artificial viscosity affects mainly the stability of the solution (and the amplitude of the ripples behind the shock); in the temperature formulation, it changes the speed of propagation of the shock significantly. For the latter equation, we find [8] that the larger the viscosity coefficient, the more the viscous heating and the better the agreement between the numerical solution and the analytic solution. However, the peak density behind the shock decreases with increasing artificial viscosity coefficient b. The density profiles cannot be taken as the only criteria of good numerical solutions.

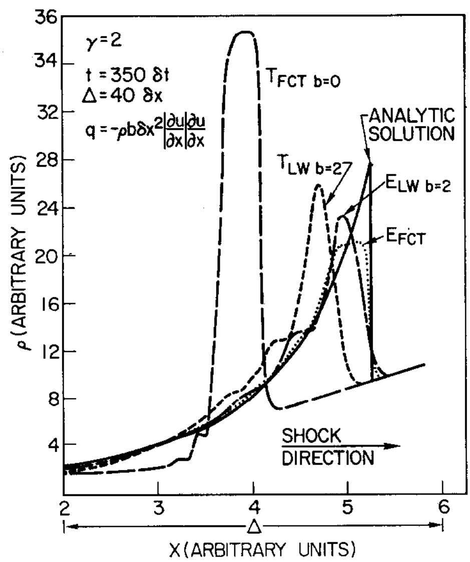

FIG. 1. Shock density profiles for shock propagating in the increasing density direction. Different energy equation formulations with flux-corrected transport (FCT) and Lax–Wendroff (LW) algorithms. The shock was located at $x = 0$ at $t = 0$.

As expected, it is found that energy conservation is better achieved by using the total energy formulation. In fact it was found that for the correct energy conserving formulation, the integrated thermal energy is increasing with time, whereas in the case of the temperature formulation it is actually decreasing. As we have pointed out, this loss of thermal energy reduces the driving force of the shock and results in the shock lagging behind the correct solution. When the temperature equation is used, energy conservation is improved by increasing the magnitude of the artificial viscosity coefficient. Because FCT allows us to run without any artificial viscosity a limiting interesting case is shown in the same figure where q has been set equal to 0 for the FCT temperature equation formulation. The shock lags behind its exact solution to a much larger extent than shown in any other case and 36 % of the total energy is lost in that extreme case.

Variations of the Results with Grid Size

Results shown previously have been obtained with a fixed grid size corresponding to a spatial resolution of 40 gridpoints per scale height. In practice, the resolution is often much coarser so the influence of the spatial resolution on the results is now investigated. Figure 2 shows the results for the density profiles when the grid size δx is multiplied by 4 so the scale height is made up of 10 gridpoints; the density profile in the shock broadens and the peak value of the density just behind the shock decreases. For this kind of spatial resolution, all equations have difficulty in simulating the presence of a strong shock and in fact look similar.

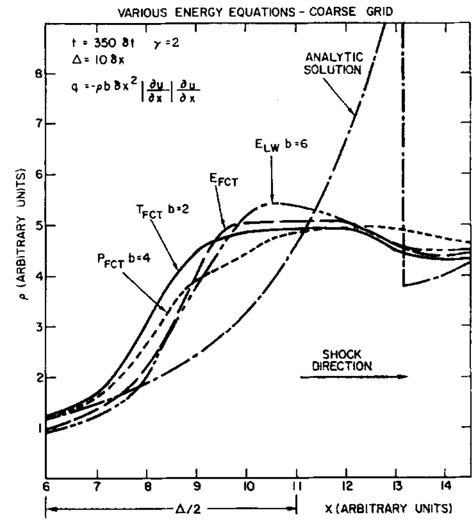

FIG. 2. Influence of grid size. Grid size is four times as large as for Figs. 1 and 3. Shock density profiles are shown for various energy equation formulations using FCT and for the total energy equation using LW (E_{LW}).

The two-cell wide flat top on the density profile is characteristic of FCT [9]. The energy equation, although showing a reduced density ratio across the shock, still approximates fairly well the shock location and yields energy conservation. The two other energy equation formulations gain energy by 10 to 16 % with the pressure equation more nearly conserving energy. Although the effect of grid size is supposed to be scaled out of the problem by the form of the artificial viscosity used, in effect when the nonconservative formulations are used energy conservation and shock location are altered by changes in grid resolution even when the same coefficient of artificial viscosity is used. Clearly, for this resolution, information about the shock has been mostly lost for both temperature and pressure equations and suggests that even 10 gridpoints per scale height with the energy equation represents a minimum resolution in order to provide a meaningful solution.

The results presented so far indicate that only the total energy equation formulation yields a correct result in which the shock location does not depend strongly on the viscosity coefficient or on changes in the grid size. For the other energy equation formulations, although it is possible to find an optimum value for the viscosity coefficient in each specific case, this value is not independent of changes in grid size or problem parameters.

RESULTS FOR DECREASING DENSITY CASE

In case of an exponentially decreasing density medium, the "analytic" self-similar solution does not exist and has to be replaced by the numerical solution of the ordinary-differential equations appearing in [5].

The specific heat ratio γ was chosen to be equal to 7/5 for this case ($\alpha = 5.45$) and results are summarized briefly below.

Figure 3 shows the density profiles for a very strong shock propagating in a decreasing density medium for the FCT and LW schemes, respectively. At the time it is shown ($t = 200\ \delta t$) the shock has traveled approximately a distance equal to 1.2Δ. The grid size is the same as that used in Fig. 1. Note that this time the shock is accelerating.

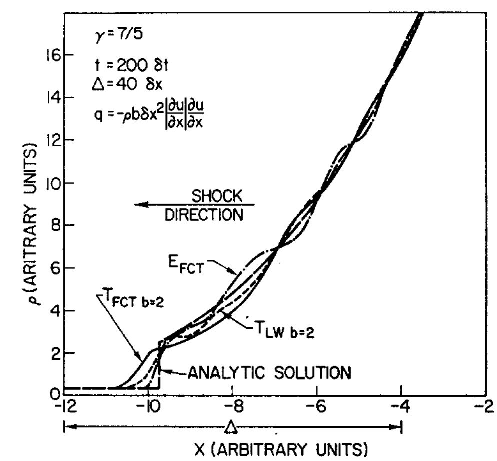

FIG. 3. Shock density profiles for shock propagating in the decreasing density direction. Temperature and energy equation formulations using FCT and LW. The shock was located at $x = 0$ at $t = 0$.

Several interesting features may be noted from this graph. First, using the same artificial viscosity coefficient as in Fig. 1 results in the shock propagating too fast for both the FCT and LW schemes using the temperature equation. This is in contrast to the results of the increasing density case shown in Fig. 1 where a coefficient $b = 2$ resulted in too slow a shock propagation.

Integrated total and thermal energies can be computed as previously. The differences are much smaller for this decreasing density case (of the order of 3 %) and this can easily be explained by the fact that since the shock propagates into a region of decreasing density, the energy carried by the shock represents a decreasingly smaller fraction of the total initial energy. Thus, the error in the shock dynamics is not reflected as much in the total energy conservation and energy conservation is a less useful check on accuracy.

CONCLUSIONS

In this study, it has been shown that the numerical results obtained for shock speed and instantaneous profile in an exponentially varying density medium can differ largely due to the choice of energy equation and spatial resolution. By comparison with an analytic solution, it has been shown that only the conservative energy equation is reliable. Even in this best case, a fairly fine spatial resolution is needed in order to derive accurate results. The total energy equation is thus superior in all respects to the nonconservative forms. Although this result is already known [4] attempts to use the nonconservative formulation with artificial viscosity for viscous heating in shocks have been made repeatedly. Further, this work has allowed us to quantify this notion for specific cases by estimating the error made when a nonconservative form is used.

The inclusion of some artificial viscosity is necessary not only for stability but to produce the necessary shock heating in the case of the temperature and pressure formulations. By suitable adjustment of the coefficient of artificial viscosity one can obtain a wide range of shock profiles and shock heating and achieve near conservation and therefore good solutions. However, it was found that there is no unique way to choose this coefficient and the precise value to achieve conservation depends both on the grid size and the nature of the problem.

The FCT algorithm does not require artificial viscosity for stability and maintains a steep profile rather independent of the value of artificial viscosity. Thus, if the temperature or pressure equation must be used, FCT gives more flexibility in achieving the correct amount of heating in the shock front. In addition, in the case of the total energy formulation, the FCT scheme requires no artificial viscosity at all, removing an eventual additional restriction on the time step.

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D. G. COLOMBANT

Science Applications, Inc.
McLean, Virginia 22101

AND

J. H. GARDNER

Plasma Physics Division
Naval Research Laboratory